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Estimates for the Boundary Values of the Solutions of Certain Nonlinear Elliptic Equations

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Nonlinear elliptic equations are considered in bounded domains. The solutions are supposed to be constant on the boundary and to have a prescribed flux. Bounds for the boundary values are constructed and isoperimetric inequalities are derived.

1. INTRODUCTION

Let $D \subset R^N$ be a bounded domain, $x = (x_1, ..., x_N)$ be a generic point in R^N, and $dx = dx_1 dx_2 \cdots dx_N$. L stands for the elliptic operator

$$L = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right)$$

and $M > 0$ is ay given number. In this paper we shall consider problems of the form

$$Lu = f(u) \text{ in } D, \qquad u = \text{constant on } \partial D, \quad \text{and} \quad \int_D f(u)\, dx = M, \tag{1.1}$$

where the boundary value of u, say $u(\partial D) =: \alpha$, is not known a priori. The following assumptions will be made on the coefficients:

(A1) $\quad a_{ij}(x) \quad$ is real analytic in $\bar{D}$

(A2) $\quad \sum_{i,j=1}^{N} a_{ij}(x)\, \xi_i \xi_j \geqslant \sum_{i=1}^{N} \xi_i^2 \quad$ for all $\xi \in R^N$ and $x \in D$.

We will require f to verify the conditions

(F1) $\quad f(t)$ is real analytic in (u_0, ∞), $\quad$ ($u_0 = -\infty$ is allowed)

(F2) $\quad f(t) > 0 \quad$ for $\quad t \in (u_0, \infty)$

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(F3) $\lim_{t\to u_0} f(t)=0, \qquad \lim_{t\to\infty} f(t)=\infty$

(F4) $f'(t)\geqslant 0 \quad \text{in} \quad (u_0,\infty).$

This type of problem appeared for the first time in a paper by Keller [4] where he derived the equilibrium condition of a uniformly charged gas in a perfectly conducting container. He also proved the existence of a unique solution, gave an iteration scheme for constructing this solution assuming the boundary value α to be known, and constructed an interesting pointwise bound for the solution, which is independent of α.

Our main purpose is to give estimates for α. We start with some known results on the existence and qualitative behavior of the solution. For convenience we shall also sketch the proofs. In the following sections we then present several methods which lead to different types of estimates. The most interesting result is the upper bound of Section 4 which depends only on the volume of D and which is isoperimetric. We compute this bound for the particular case of an ideal gas where $L=\Delta$ and $f(t)=\lambda e^t$, and derive a further isoperimetric inequality based on the special form of f. Finally we indicate a variational formulation of problem (1.1) which seems to be especially suitable for treating linear problems.

2. Preliminary Results

Lemma 2.1. *The problem $Lv=f(v)$ in D, $v=v_0$ on ∂D has for given $v_0\geqslant u_0$ a unique solution $u_0\leqslant v(x)\leqslant v_0$.*

Proof. According to a result of [1] it suffices to construct two functions $\varphi(x)\leqslant\psi(x)$ satisfying $L\varphi\geqslant f(\varphi)$ in D, $\varphi\leqslant v_0$ on ∂D, and $L\psi\leqslant f(\psi)$ in D, $\psi\geqslant v_0$ on ∂D. Set $\psi(x)=v_0$ and $\varphi(x)=u_0$ if u_0 is finite; otherwise take for φ the solution of the boundary value problem $L\varphi=f(v_0)$ in D, $\varphi=v_0$ on ∂D. By the maximum principle we have $\varphi(x)\leqslant v_0$. The proof of the first part is thus established.

In order to prove the uniqueness we suppose that v_1 and v_2 are two different solutions. Let $D^+ := \{x\in D: v_1(x)>v_2(x)\}$. Because of the boundary condition, the difference $\delta=v_1-v_2$ vanishes on ∂D^+. It satisfies $L\delta\geqslant 0$ in D^+ and achieves therefore its maximum on ∂D^+. Hence $\delta<0$ in D^+, which is a contradiction to our assumption. Consequently D^+ is empty and $v_1(x)\leqslant v_2(x)$. Similarly we show that $v_1(x)\geqslant v_2(x)$. This completes the proof of the lemma.

From the maximum principle we get in addition:

LEMMA 2.2. *Let v and v' be the solutions of $Lw = f(w)$ in D with $w = v_0$ or $w = v_0'$, respectively, on ∂D. If $v_0 \geqslant v_0'$, then $v(x) \geqslant v'(x)$.*

Furthermore we have

LEMMA 2.3. *The solution v depends continuously on v_0.*

Proof. Let $\varepsilon > 0$ be any small number. In view of (F4) and the previous lemma, the difference $d(x) := v(x : v_0 + \varepsilon) - v(x : v_0)$ satisfies $Ld \geqslant 0$ and according to Hopf's maximum principle takes its maximum at the boundary. Hence $0 < d(x) < \varepsilon$ in D, which proves the assertion.

The existence of a solution of (1.1) follows immediately from

LEMMA 2.4. *Consider $M = M(\alpha)$ as a function of the boundary value α. Then $M(\alpha)$ is continuous, monotone and $\lim_{\alpha \to u_0} M(\alpha) = 0$, $\lim_{\alpha \to \infty} M(\alpha) = \infty$.*

Proof. The first statements follow directly from Lemmas 2.2 and 2.3, and $\lim_{\alpha \to u_0} M(\alpha) = 0$ is due to (F3). In order to show that $M(\alpha)$ is unbounded as $\alpha \to \infty$, we first note that by Green's theorem

$$\alpha M(\alpha) = \int_D \sum_{i,j=1}^{N} \left\{ a_{ij} \left(\frac{\partial u}{\partial x_i} \right) \left(\frac{\partial u}{\partial x_j} \right) \right\} dx + \int_D uf(u)\, dx. \tag{2.1}$$

From (A2) we get, setting $\mathscr{D}(u) := \int_D \operatorname{grad}^2 u\, dx$,

$$M(\alpha) \geqslant \alpha^{-1} \mathscr{D}(u) + \alpha^{-1} \int_D uf(u)\, dx \qquad \text{for} \quad \alpha > 0. \tag{2.2}$$

Since for $\alpha > 0$, u is bounded from below by the solution of $Lv = f(0)$ in D, $v = 0$ on ∂D, the second term on the right-hand is bounded from below by c_0/α, where c_0 is independent on α. Two possibilities can now occur, namely either $\alpha^{-1}\mathscr{D}(u)$ is unbounded for $\alpha \to \infty$, or it is bounded. In the first case it follows from the previous remark that $M(\alpha) \to \infty$ as $\alpha \to \infty$. Let $\alpha^{-1}\mathscr{D}(u) < c_1$ for $\alpha \geqslant \alpha_0$. By the trace theorem [5] there exists a constant $c_2 > 0$ independent of α such that

$$\alpha^2 \oint_{\partial D} ds \leqslant c_2 \left\{ \mathscr{D}(u) + \int_D u^2\, dx \right\},$$

and in view of our assumption $\alpha^{-1}\mathscr{D}(u) < c_1$ for $\alpha \geqslant \alpha_0$,

$$\alpha^2 \oint_{\partial D} ds \leqslant c_2 \left\{ \alpha c_1 + \int_D u^2\, dx \right\} \qquad \text{for all } \alpha \geqslant \alpha_0.$$

Since

$$\int_D u^2\,dx < \varepsilon^2\alpha^2 \int_{u\leqslant\varepsilon\alpha} dx + \alpha^2 \int_{u>\varepsilon\alpha} dx, \qquad \text{for all } \varepsilon\in(0,1)$$

it follows that

$$\alpha^2 \oint_{\partial D} ds \leqslant c_2 \left\{\alpha c_1 + \alpha^2\varepsilon^2 \int_{u\leqslant\varepsilon\alpha} dx + \alpha^2 \int_{u>\varepsilon\alpha} dx\right\}.$$

Hence there exist numbers $\varepsilon_0>0$, α_1 and c_3 independent of α such that

$$\int_{u>\varepsilon_0\alpha} dx \geqslant c_3 \qquad \text{for all } \alpha\geqslant\alpha_1.$$

The last inequality together with (2.2) implies

$$M(\alpha)\geqslant \varepsilon_0 f(\varepsilon_0\,\alpha)\,c_3,$$

and thus $M(\alpha)\to\infty$ for $\alpha\to\infty$.

3. Method of Upper and Lower Solutions

We start with a definition which is opposite to the standard one, but which is more suitable for our purpose.

Definition. A function $\varphi\in C^2(D)\cap C^0(\bar D)$ with values in $[u_0,\infty)$ is called a *lower solution* if

(i) $L\varphi\leqslant f(\varphi)$ in D,

(ii) $\int_D f(\varphi)\,dx\leqslant M$.

It is called an *upper solution* if the inequality signs are reversed.

Then we have:

Theorem 3.1.

(i) *If φ is a lower solution, then $\alpha\geqslant\min_{x\in\partial D}\varphi(x)$.*

(ii) *If ψ is an upper solution, then $\alpha\leqslant\max_{x\in\partial D}\psi(x)$.*

Proof. (i) Suppose that $\alpha<\min_{x\in\partial D}\varphi(x)$. Put $d(x):=\varphi(x)-u(x)$. Because of our assumption, the region $D^+=\{x\colon\{x\colon d(x)>0\}$ is not empty. The same is true for $D^-:=\{x\colon d(x)\leqslant 0\}$. Since $d>0$ on ∂D, ∂D^- is entirely contained in D. In D^- we have $Ld\leqslant 0$. By Hopf's maximum principle

$d(x) > \min_{x \in \partial D} d(x) = 0$ in D^-, contradicting the definition of D^-. Hence $\alpha \leqslant \min_{x \in \partial D} \varphi(x)$.

Exactly in the same way we prove the second assertion.

EXAMPLES. (1) Take $\varphi(x) \equiv c$ where c is determined such that $f(c) \int_D dx = M$. The constant c is a lower solution and by Theorem 3.1(i) we have $\alpha \geqslant c$.

(2) Consider problem (1.1) with $L = \Delta$, and put ψ for the radially symmetric solution of $\Delta\psi = f(\gamma)$, satisfying $\psi \leqslant \gamma$ on ∂D. It is of the form

$$\psi = \gamma - f(\gamma)\{r_0^2 - |x|^2\}/(2N), \qquad r_0 := \sup_{x \in \partial D} |x|.$$

If we can find a number γ such that

$$\int_D f(\psi)\, dx = \int_D f[\gamma - f(\gamma)\{r_0^2 - |x|^2\}/(2N)]\, dx = M,$$

then $\alpha \leqslant \gamma$.

(3) Again let $L = \Delta$ and let D be contained between the hyperplanes $x_1 = a$ and $x_1 = b$. Consider in D the function $v = v(x_1)$ satisfying $v'' = f(v)$ in (a, b), $v(a) = v(b) = \beta$. If $\int_D f(v)\, dx \geqslant M$, then $\alpha \leqslant \beta$. Otherwise we have $\alpha \geqslant \min_{x \in \partial D} v \geqslant v((a+b)/2)$.

4. DOMAINS WITH THE SAME VOLUME

In this section we make use of the level surface technique [2, 6]. Besides problem (1.1) let us introduce the particular problem

$$\Delta u^* = f(u^*) \quad \text{in } D^*, \quad u^* = \alpha^* \quad \text{on } \partial D^* \quad \text{and} \quad \int_{D^*} f(u^*)\, dx = M, \tag{4.1}$$

where D^* is the sphere of the same volume as D centered at the origin and α^* is an appropriate constant. Our main result states

THEOREM 4.1. *Under the assumptions of Section 1 we have $\alpha \leqslant \alpha^*$. The equality sign holds if and only if $L = \Delta$ and $D = D^*$.*

Proof. We shall put $D(\mu) := \{x: u(x) \leqslant \mu\}$, where μ varies in the interval $[u_{\min}, \alpha]$. Moreover we write $a(\mu)$ for the volume of $D(\mu)$. This function is monotone with $a(u_{\min}) = 0$ and $a(\alpha) = A$. Because of our regularity assumptions, its inverse $\mu(a)$ exists and is Lipschitz [2]. The following relation holds a.e. [2]:

$$\frac{d\mu}{da} = \left\{ \oint_{\partial D(\mu)} ds/|\operatorname{grad} u| \right\}^{-1}, \tag{4.2}$$

ds being the surface element on $\partial D(\mu)$. Schwarz's inequality and the isoperimetric inequality imply

$$\left(\frac{d\mu}{da}\right)^{-1} \oint_{\partial D(\mu)} |\operatorname{grad} u|\, ds \geqslant \left\{ \oint_{\partial D(\mu)} ds \right\}^2 \geqslant q(a), \tag{4.3}$$

where $q^{1/2}(a)$ is the surface area of an N-dimensional sphere of volume a. Here we have used the fact that the volume of $D(\mu)$ is a. From the Green's identity, using the fact that u is constant on $\partial D(\mu)$, we get

$$\int_{D(\mu)} Lu\, dx = \oint_{\partial D(\mu)} \left(\sum_{i,j=1}^{N} a_{ij} n_i n_j \right) |\operatorname{grad} u|\, ds,$$

$$n = (n_1, \ldots, n_N) \quad \text{outer normal}$$

and from (A2) we conclude that

$$\int_{D(\mu)} f(u)\, dx = \int_{D(\mu)} Lu\, dx \geqslant \oint_{\partial D(\mu)} |\operatorname{grad} u|\, ds.$$

Since $\int_{D(\mu)} f(u)\, dx = \int_0^a f(\mu(\beta))\, d\beta$, it then follows that

$$\int_0^a f(\mu)\, d\beta \geqslant \mu'(a)\, q(a) \qquad \text{in} \quad (0, A). \tag{4.4}$$

Equality holds for all a if and only if

(i) $L = \Delta$,

(ii) all $D(\mu)$ are spheres.

This is the case for problem (4.1). Let us put μ^* for μ, if u is replaced by u^*. According to the previous remark

$$\int_0^a f(\mu^*)\, d\beta = \mu^{*\prime}(a)\, q(a) \qquad \text{in} \quad (0, A). \tag{4.5}$$

Our next step is to compare $\mu(A)$ with $\mu^*(A)$. For this purpose let $\delta(a) := \mu(a) - \mu^*(a)$. Since $\int_D f(u)\, dx = \int_{D^*} f(u^*)\, dx$, $\delta(a)$ must change sign. Moreover, it follows from (4.4) and (4.5) that $\delta'(A) \leqslant 0$. Our proof is completed if we can show that $\delta(A) \leqslant 0$. Let us assume the opposite,

$\delta(A) > 0$. It is clear from the last observations that there exists a value $A_0 \leqslant A$ such that $\delta(A_0) > 0$, $\delta'(A_0) = 0$, and $\delta(a) > 0$ in (A_0, A). If D is not a sphere, the case $A_0 = A$ can be excluded. From (4.4) and (4.5) we have

$$0 = \delta'(A_0) \leqslant q^{-1}(A_0) \int_0^{A_0} \{f(\mu) - f(\mu^*)\}\, d\beta. \tag{4.6}$$

Put for the moment $M(a) := \int_0^a f(\mu)\, d\beta$ and $M^*(a) := \int_0^a f(\mu^*)\, d\beta$. Then (4.6) implies

$$M(A_0) \geqslant M^*(A_0). \tag{4.7}$$

This result together with $f(\mu(a)) > f(\mu^*(a))$ in (A_0, A) yields $M = M(A) > M^*(A)$. Thus $\delta(A)$ cannot be positive.

A further consequence of the last proof is

THEOREM 4.2. *Under the same assumptions as for Theorem* 4.1 *we have* $u_{\min} \geqslant u^*_{\min}$.

Proof. Suppose that $\delta(0) < 0$. Since $\delta(a)$ changes sign there exists a value A_1 such that $\delta(A_1) = 0$, $\delta'(A_1) \geqslant 0$, and $\delta(a) < 0$ in $(0, A_1)$. From (4.4) and (4.5) we then get

$$M(A_1) \geqslant M^*(A_1). \tag{4.8}$$

On the other hand, we have $f(\mu(a)) \leqslant f(\mu^*(a))$ in $(0, A_1)$ and consequently $M(A_1) < M^*(A_1)$, contradicting (4.8). Hence $\delta(0) = u_{\min} - u^*_{\min} \geqslant 0$.

Complement. Hopf's maximum principle guarantees that $|\operatorname{grad} u| \neq 0$ on ∂D provided that u is differentiable up to the boundary and that ∂D is of class C^2. By virtue of (4.2) $\{\mu'(A)\}^{-1} = \oint_D ds/|\operatorname{grad} u|$, and by (4.4) we have $M/q(A) \geqslant \mu'(A)$. Combining these inequalities we obtain

$$q(A)^{-1} M \oint_{\partial D} ds \geqslant \min_{x \in \partial D} |\operatorname{grad} u|. \tag{4.9}$$

The equality sign holds again only for the sphere.

5. SPECIAL CASE

We shall be concerned with the following special case of (1.1):

$$\Delta u = \lambda e^u \quad \text{in} \quad D \in R^2, \qquad u = \alpha \quad \text{on} \quad \partial D,$$

$$\lambda \int_D e^u\, dx = M, \qquad \lambda > 0 \text{ given number}. \tag{5.1}$$

This is the problem of a charged ideal gas in a cylinder of cross-section D. M corresponds to the mass per unit height [4].

EXAMPLE. *In* $D^* = \{x: |x| < R\}$, the solution of (5.1) can be calculated explicitely, namely

$$u^*(x) = \log b \left(1 - \frac{\lambda}{8} b|x|^2\right)^{-2},$$

$$\alpha^* = \log b \left(1 - \frac{\lambda}{8} bR^2\right)^{-2}, \qquad \text{where} \quad M = \frac{\pi\lambda bR^2}{1 - \lambda bR^2/8}.$$

In the light of Theorems 4.1 and 4.2 comparing this result with the solution of (5.1) shows that

$$e^{\alpha} \leqslant \frac{M^2 + 8\pi M}{8\pi\lambda A} \qquad \text{and} \qquad e^{u_{\min}} \geqslant \frac{8\pi M}{\lambda A(M + 8\pi)}. \tag{5.2}$$

A simple lower for α is given in Example 1 of Section 3, namely

$$e^{\alpha} \geqslant M/(\lambda A). \tag{5.3}$$

In [3] we have constructed for simply connected domains the estimate

$$e^{\alpha} \geqslant \frac{M^2 + 8\pi M}{2\lambda l^2}, \qquad l := \oint_{\partial D} ds. \tag{5.4}$$

The equality sign holds again only for the circle.

Although (5.3) is not isoperimetric it furnishes for small M better results than (5.4).

We now indicate a further bound using *Rellich's identity* [7, 2]. For this purpose let D be a *starlike* domain, i.e., every straight line emanating from the origin of the coordinate system intersects D in exactly one point. Then the scalar product between the outer normal n at $x \in \partial D$ and the vector x, say (n, x), is positive everywhere. Rellich's identity applied to the solution of (5.1) gives

$$\oint_{\partial D} \frac{(n, x)}{2} \left(\frac{\partial u}{\partial n}\right)^2 ds - \lambda e^{\alpha} \oint_{\partial D} (n, x)\, ds + 2\lambda \int_D e^u\, dx = 0. \tag{5.5}$$

Put $B := \oint_{\partial D} (n, x)^{-1}\, ds$. Schwarz's inequality yields

$$\oint_{\partial D} (n, x) \left(\frac{\partial u}{\partial n}\right)^2 ds \geqslant B^{-1} \left\{\oint_{\partial D} \frac{\partial u}{\partial n} ds\right\}^2 = B^{-1} M^2.$$

In addition, we have $\oint_{\partial D}(n, x)\,ds = 2A$. Introducing these relations into (5.5), we find

$$(M^2/2B) + 2M - 2\lambda A e^{\alpha} \leqslant 0. \tag{5.6}$$

This proves

THEOREM 5.1. *For any starlike domain the boundary value α of* (5.1) *satisfies*

$$e^{\alpha} \geqslant (M^2 + 4BM)/(4\lambda AB).$$

Equality holds for the circle.

It should be noticed that this estimate is always better than (5.3).

6. CONNECTIONS WITH THE CALCULUS OF VARIATION.

Let us again consider problem (1.1). In order to simplify the presentation we take $L = \Delta$. The extension to the general case is immediate and will therefore be omitted. Let $F(t)$ be any function such that $F'(t) = f(t)$, and define for any piecewise continuously differentiable function v which is constant on ∂D, the *energy*

$$J[v] := \mathscr{D}(v) + 2\int_D F(v)\,dx - 2Mv(\partial D).$$

LEMMA 6.1. *If u solves* (1.1), *then $J[v] \geqslant J[u]$ for any v defined above.*

Proof. Let $v = u + h$. Using Taylor's expansion formula, we find

$$J[u+h] = \mathscr{D}(u) + 2\int_D \operatorname{grad} u \operatorname{grad} h\,dx + \mathscr{D}(h) + 2\int_D F(u)\,dx$$
$$+ 2\int_D f(u)\,h\,dx + \int_D f'(\tilde{u})\,h^2\,dx - 2M(u(\partial D) + h(\partial D)).$$

where $\tilde{u}$ lies between u and v. Since

$$\int_D \operatorname{grad} u \operatorname{grad} h\,dx = -\int_D h\,\Delta u\,dx + \oint_{\partial D} h\frac{\partial u}{\partial n}\,ds,$$

we have

$$J[u+h] = J[u] + \mathscr{D}(h) + \int_D f'(\tilde{u})\, h^2\, dx \geqslant J[u]. \tag{6.1}$$

The proof of the lemma is thus completed.

In the linear case $f(t) = t + b$ there is a particularly simple relation between α and $J[u]$. Namely, if we set $F(t) = t^2/2 + bt$, we get

$$J[u] = -\alpha M + b \int_D u\, dx.$$

In view of the condition $M = \int_D (u+b)\, dx$, we conclude that

$$J[u] = -\alpha M + bM - b^2 A. \tag{6.2}$$

Lemma 6.1 together with (6.2) yields for any piecewise differentiable function v with $v = \text{constant}$ on ∂D

$$\alpha M \geqslant bM - b^2 A - J[v].$$

Moreover, we obtain the following monotonicity result for the boundary value $\alpha(b)$ considered as a function of b.

LEMMA 6.2. *If $b_2 \leqslant b_1$, we have*

(i) $\alpha(b_2) \geqslant \alpha(b_1)$ *if* $M \leqslant (b_1 + b_2) A$,

(ii) $\alpha(b_2) \leqslant \alpha(b_1)$ *if* $M \geqslant (b_1 + b_2) A$.

Equation (6.2) illustrates also the behavior of α under Steiner symmetrization [6]. Then D is transformed into a domain D' of the same volume which is symmetric with respect to $x_1 = 0$, and u is changed into a function u', being symmetric with respect to $x_1 = 0$. For further details and precise statements we refer to [6]. By virtue of the well-known properties of this symmetrization, it follows that $J_{D'}[u'] \leqslant J_D[u]$, and in view of Lemma 6.1 and (6.2) the boundary value α' corresponding to D' is greater than or equal to α. In other words we have shown the

LEMMA 6.3. *Let $f(t) = t + b$, $L = \Delta$ and consider $\alpha = \alpha(D)$ as a function of D. Then $\alpha(D)$ increases under Steiner symmetrization.*

Remark. This lemma does not extend directly to the general operator L.

We conclude mentioning two

SPECIAL CASES. (1) Among all triangles of given area, the equilateral one gives the largest value of α.

(2) Among all plane rectangles of given area, the square yields the largest value of α.

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